SOME SYMMETRIC SYSTEMS OF MINIMAL SURFACES

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ABSTRACT

In this paper we use a method of an earlier paper in order to prove the existence of some symmetric systems of minimal surfaces bounded by curves having self-intersections.

In a previous paper [8] a mathematical existence proof was given for a system of three surfaces of least area spanning three rectifiable Jordan arcs which are joined at two points, subject to additional symmetry conditions (see Fig. 1). The proof was based on the solution of a variational problem for a Dirichlet functional in a suitable class of functions, and obtained by (a) solving a Riemann Hilbert problem, and (b) a method similar to that used by H. Lewy in [4] for the solution of a free boundary problem for a minimal surface.

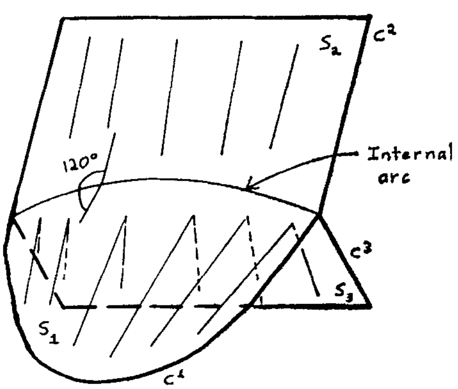

Fig. 1. System of three surfaces.

Systems of minimal surfaces similar to that of [8] occur for many different contours, and in many natural situations. Striking examples are given in [3] and [9]. In this paper we extend the existence proofs of [8] to two such contours.

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The first is that of Fig. 2 consisting of two Jordan curves C^1, C^2 which are reflections of each other with respect to a plane $\mathscr{P}$. A soap-film experiment of the Plateau type for such a contour yields a catenoidal-like system of three surfaces, two of them

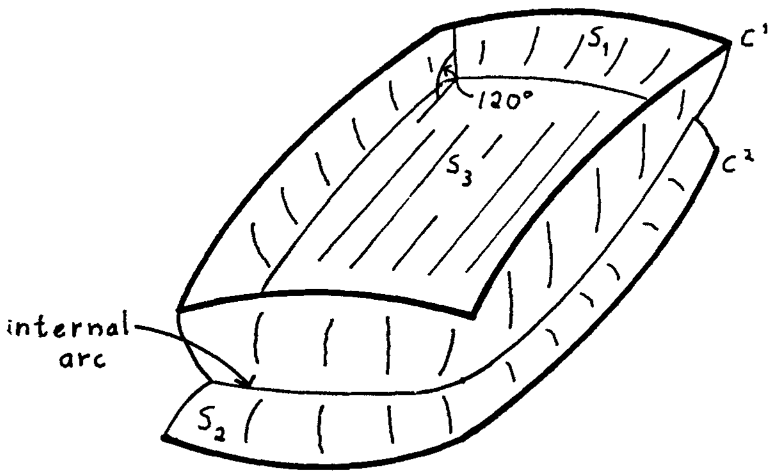

Fig. 2. System of three surfaces.

simply connected and the third lying in $\mathscr{P}$, and bounded by a free internal boundary. (See Fig. 2). In Section 1 we give a corresponding mathematical existence proof for a rectifiable curve b in $\mathscr{P}$ and surfaces S_1, S_2, S_3 spanning bC^1, bC^2 and b, respectively, with S_1, S_2 doubly connected, having least surface area among all such systems. The second contour (in Fig. 3) consists of a rectifiable Jordan curve C, symmetric with respect to a plane $\mathscr{P}$ which it intersects at points P and Q, together with a curve C' in $\mathscr{P}$ originating at P and passing "through" C. For this contour, Plateau's experiments yield a surface system having an additional free

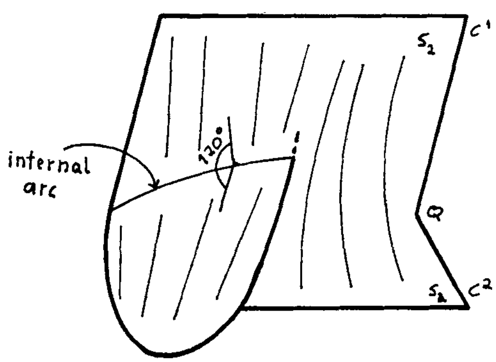

Fig. 3. A surface with a free point.

point of intersection of an internal arc and the curve C', as in Fig. 3. In Section 2 we solve the corresponding mathematical existence problem, proving the existence of an internal arc b' joining P to a point P^* of C', and surfaces S_1 bounded by b' and the arc PP^* of C', and S, spanning b' and C, such that a) the area of this surface is least among all such surface systems; b) b' is analytic and rectifiable; c) the angle of intersection of S_1 and the portions of S above and below $\mathscr{P}$ is 120°. As in [8] the proof of the least area property requires certain additional inequality assumptions.

Throughout this discussion we use the notation of [8]. In particular $A(S)$ denotes the area of a surface S, and $D_K(\bar{x}) = \frac{1}{2}\int\int_K(\bar{x}_u^2 + \bar{x}_v^2)dudv$ the Dirichlet integral of a vector function $\bar{x}(u,v)$ over a domain K of the u, v plane. For any point set W, we denote the closure of W by $\bar{W}$.

1. The Catenoidal System. Let C^1, C^2 be rectifiable Jordan curves in x, y, z space which are reflections of each other with respect to the plane $\mathscr{P}: z = 0$. A surface system spanning these curves is "admissible" if it consists of a rectifiable Jordan curve b in $\mathscr{P}$ together with doubly connected surfaces S_1, S_2 spanning bC^1, bC^2 which are reflections of each other with respect to $\mathscr{P}$, and a surface S_3 $\mathscr{P}$ spanning b.

The contour cannot be spanned by a system of minimal surfaces unless C^1, C^2 are sufficiently close. (see [7]) Thus we make the following assumption corresponding to the sufficient condition of J. Douglas for the Plateau-Douglas problem: ([1], chap. IV).

Let d be the area of a minimal surface of least area spanning C^1, and d' the greatest lower bound of the sufcace areas of all admissible systems spanning C^1, C^2. Then

$$d' < 2d. \tag{1}$$

We wish to prove the existence of an admissible surface system of least area; this is done by solving a variational problem for a Dirichlet functional (in Theorem 1) and referring to the results of [8] sec. 7, for the least area property.

For $0 < r < 1$, let K_r denote the open annulus of the u, v plane $r^2 < u^2 + v^2 < 1$, bounded by the unit circle L and the concentric circle $L_r: u^2 + v^2 = r^2 < 1$ of radius r. (see Fig. 4). Let R be the convex hull of the projection of C^1 onto $\mathscr{P}$. By (1) there exists a rectifiable Jordan curve b in R, and a doubly connected

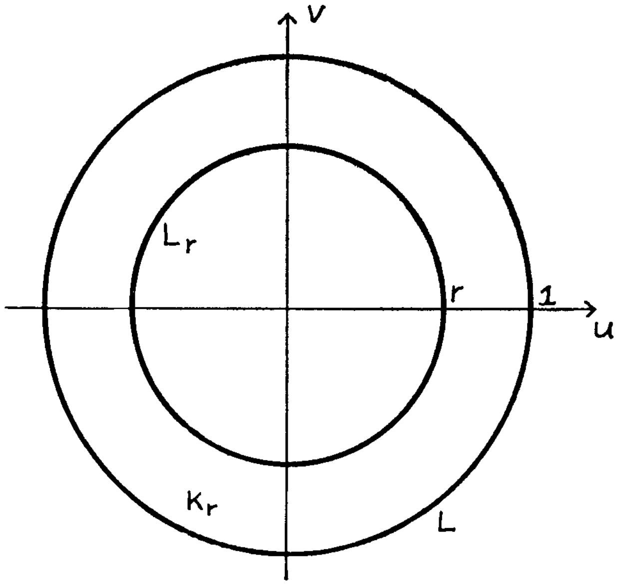

Fig. 4.

minimal surface S_1 spanning bC^1 and represented on K_r for some r depending on b by a harmonic and isometric vector $\bar{x}(u,v) = (x(u,v), y(u,v), z(u,v))$. (See [1]). Let S_2 be the reflection of S_1 with respect to $\mathscr{P}$ (spanning bC^2) and S_3 the plane region bounded by b. The surface area of this system is

$$\sum_{i=1}^{3} A(S_i) = F(\bar{x}) = 2D_{K_r}(\bar{x}) + \int_{L_r} x(s)y'(s)ds. \tag{2}$$

Definition 1. We say that a function $\bar{x}(u,v) = (x(u,v),\ y(u,v), z(u,v))$ is in the class $\mathscr{F}$ if

a) $\bar{x}$ maps L monotonically onto C^1;
b) For some $r > 1$, $\bar{x}$ is continuous on $\overline{K_r}$ and harmonic on K_r;
c) $D_{K_r}(\bar{x}) < \infty$;
d) $z = 0$ on L_r.

We pose the

Minimum Problem 1. Minimize F over $\mathscr{F}$.

As in [8], sec. 3, one can prove the following lemma:

Lemma 1. *F can be defined for all $\bar{x} = (x,y,z)$ in $\mathscr{F}$, and even for those $\bar{x}$*

for which x, y are not defined on L_r. Moreover $F(\bar{x}) \geqq m$, for some m independent of $\bar{x}$.

Using either of the methods indicated in [8] one can prove the following basic theorem:

THEOREM 1. *Minimum Problem 1 has a solution.*

As in [1] we replace a minimizing sequence for F in $\mathscr{F}$ by a sequence $\{\bar{x}^n\}$ where (a) *$\bar{x}^n$ is continuous on a domain $\overline{K_{r_n}}$, harmonic on K_{r_n}, analytic on L, and* (b) *$\bar{x}^n$ converges on L to a monotonic representationof C^1 as $n \to \infty$. As in* [1], *condition* (1) *implies that such a sequence is cohesive, or that there is a number $\delta > 0$, independent of n, such that each closed curve on the surface defined by $\bar{x}^n$ having diameter less than δ can be continuously contracted to a point of the surface; this guarantees the non-degeneration of our surface system, as well as the convergence of the domains K_{r_n} to an annulus K_r, $0 < r < 1$.*

For minimum problem 1 the transversality condition is seen to be

(3) $$y_s^n - 2x_v^n,\ 2y_v^n + x_s^n \to 0 \text{ on } L_{r_n},$$

where $\partial/\partial v$, $\partial/\partial s$ denote differentiation in the radial direction away from the origin and in the clockwise tangential direction on L_{r_n}, respectively. For a solution $\bar{x} = (x, y, z)$ of the minimum problem

(4) $$y_s - 2x_v = 2y_v + x_s = 0,$$

implying as in [8] that the three component surfaces of the system of least area meet along the interior arc at 120° angles.

A solution to the minimum problem now follows directly from the theorem:

THEOREM 2. *Let a(s), c(s) be analytic on L, and $0 < t < 1$. Then there exist functions A(u, v), C(u, v) on K_t, which agree with a, c on L and obey the following conditions:*

1) *A, C are continuous on $\overline{K_t}$, harmonic on K_t, and have finite Dirichlet integrals;*

2) *On L_t, A and C are analytic and*

(5) $$C_s - 2A_v = 2C_v + A_s = 0.$$

3) *Let a^n, c^n converge uniformly to a, c on L; furthermore, let A^n, C^n coincide with a^n, c^n on L, and satisfy* 1), 2). *Then A^n, C^n converge uniformly to functions A, C which obey conditions* 1), 2).

To prove this we represent A, C in terms of a, c as in [8], using methods of Plemelj and Muskhelishvili. Let A^*, C^* be harmonic conjugates of the desired functions A, C. Defining X^*, Y^*, and their harmonic conjugates X, Y by

$$X = 2A - C^*, \quad Y = -2C - A^*,$$
$$X^* = 2A^* + C, \quad Y^* = -2C^* + A, \tag{6}$$

the conditions $A = a$, $C = c$ on L and (5) take the form

$$2X - Y^* = f = 3a, \quad 2Y + X^* = g = -3c \text{ on } L, \tag{7}$$

$$X^* = Y^* = X_\nu = Y_\nu = 0 \text{ on } L_t. \tag{7a}$$

Setting $F = X + iX^*$, $G = Y + iY^*$, $H = \begin{pmatrix} F \\ G \end{pmatrix}$, $\bar{g} = \begin{pmatrix} f \\ g \end{pmatrix}$,

and using the methods of [8] and [5], we obtain

$$H(w) = \tfrac{1}{4}(\phi(w) + \overline{\phi(1/\bar{w})} + \phi(t^4/w) + \overline{\phi(t^4/\bar{w})}); \tag{8}$$

here $\phi(w)$ is the solution to the Riemann-Hilbert problem

$$2\,\mathrm{Re}\,\phi - B\,\mathrm{Im}\,\phi = \bar{g}, \text{ on } L_{t^2}$$
$$2\,\mathrm{Re}\,\phi + B\,\mathrm{Im}\,\phi = \bar{g}, \text{ on } L, \tag{9}$$

with $w = u + iv$,

$$B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

given by

$$\phi(w) = \frac{1}{2\pi i} \int_{L+L_{t^2}} \frac{T(w)T^+(\tau)^{-1}\bar{g}(\tau)}{t - w} d\tau, \tag{10}$$

where T is a piecewise constant matrix function (see Appendix), and T^+ denotes the limiting value of T from within the annular domain. The proof of Theorem 2 (and the solution of Minimum Problem 1) now follow directly from (10), as in [8], sec. 5.

An alternate solution of the variational problem is given as follows: (see [8], sec. 6). One broadens $\mathscr{F}$ to include those functions which are not necessarily defined on L_t, but whose third component function vanishes there. By the cohesiveness assumption a minimizing sequence for F in $\mathscr{F}$ and a corresponding family of domains K_{r_n} is compact, and contains a subsequence which converges to a harmonic and isometric function $\bar{x} = (x, y, z)$ defined on a domain K_r, with $z = 0$ on L_r. If z^* is the harmonic conjugate of z on K_r, the function $H(u, v) = z(u, v) + iz^*(u, v)$ maps some annular neighborhood of L_r onto a strip of finite length in the z, z^* plane, since z^* is not single valued on K_r. (See [6], p. 34). The proof now follows in exactly the same manner as [8].

Using the methods of [8], sec. 4, one easily sees that the solution of the minimum problem defines on L_r an analytic and rectifiable curve which does not intersect itself.

2. A surface with a free point. Let C^1, C^2 be rectifiable Jordan arcs which are reflections of each other with respect to the plane $\mathscr{P}: z = 0$ and which intersect $\mathscr{P}$ at their common end points $P = (p, 0, 0)$, $Q = (q, 0, 0)$, $q < p$; suppose their projections upon $\mathscr{P}$ lie in the half-plane $y \leqq 0$. Moreover, assume that C^1, C^2 have tangents at P, Q which do not lie in $\mathscr{P}$. Let C^3 be a curve given by $y = f(x)$, with $f(p) = 0$, $f''(x) \leqq 0$, and $f(p^*) = 0$, where $q < p^* < p$. (See Fig. 5). Consider the class of all surface systems S_1, S_2, S_3 with an internal arc b lying in $\mathscr{P}$, joining P, Q and intersecting C^3 at some point P'; suppose further that S_1, S_2 span bC^1, bC^2 respectively, and are reflections of each other with respect to $\mathscr{P}$ while S_3 is in $\mathscr{P}$, bounded by C^3 and the arc of b joining P' and P. We make the additional plausible assumption that the arc of b joining P' to P lies below C^3. We seek a system of this type having least surface area among all such systems.

This problem is again solved by considering a variational problem for a Dirichlet functional in a suitable class of functions, and applying the methods of [8], sec. 7.

Let K^+ be the unit semi-disk $u^2 + v^2 < 1$, $v > 0$ of the u, v plane with upper arc L^+ and base B. As in [8], the solution surface is given by a vector function $\bar{x}(u, v) = (x(u, v), y(u, v), z(u, v))$ on K^+ such that (see Fig, 6)

1) $\bar{x}$ maps $\overline{L^+}$ onto C^1 and sat·sfies a 3-points condition there;

2) $\bar{x}$ is continuous on $\overline{L^+} + K^+$, harmonic on K^+, and $D_{K^+}(\bar{x}) < \infty$;

3) There exists a point $(u_0, 0)$ on B such that $\bar{x}(u, v)$ tends to a point $P' = (p', q')$ of C^3 as (u, v) tends to $(u_0, 0)$;

4) Among all functions satisfying 1)–3), $\bar{x}$ minimizes the functional

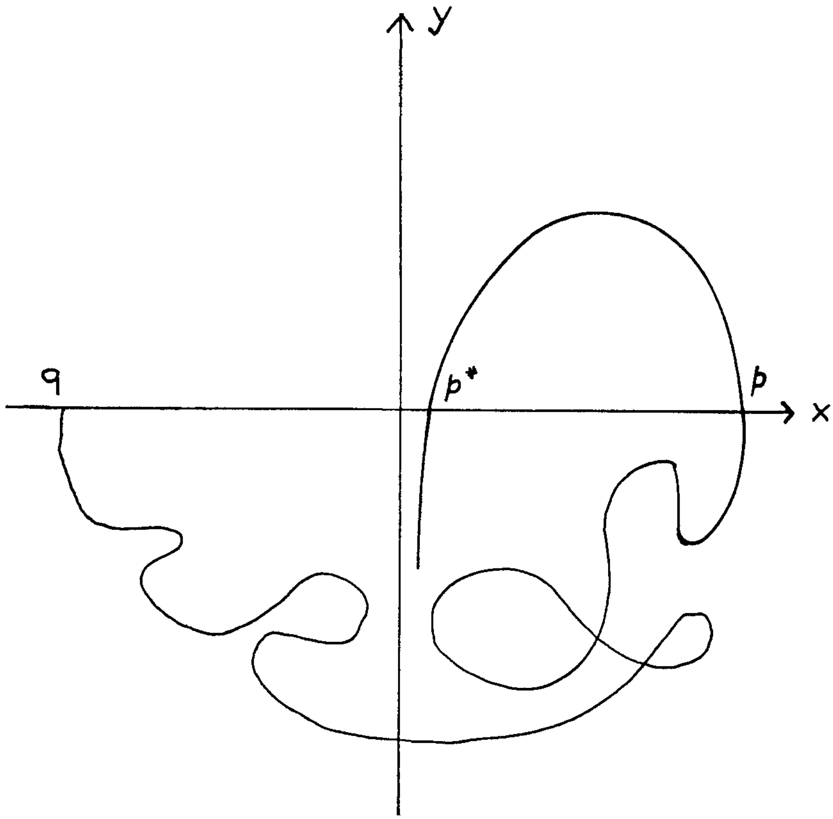

Fig. 5.

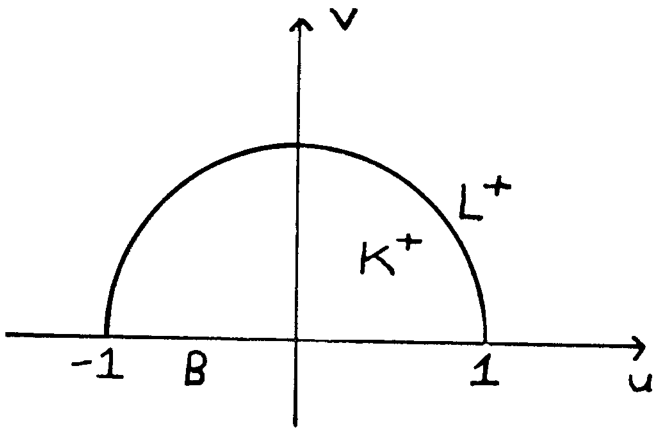

Fig. 6.

$$F(\bar{x}) = 2D_{K^+}(\bar{x}) + \int_{u_0}^{1} xy_u du + \int_{q'}^{p} f(x)dx.$$

It is seen that the sum of the last two integrals represents the area of the plane region S_3, while since $\bar{x}$ is isometric, $F(\bar{x})$ is the area of the system of surfaces defined by $\bar{x}$.

As in [8], $F(\bar{x})$ exists for all such functions $\bar{x}$, and has a lower bound independent of $\bar{x}$. Let $\{\bar{x}^n\}$ be a minimizing sequence for F in this function class, uniformly convergent on $\overline{L^+}$. Tben $\bar{x}^n$ maps some point $(u_n, 0)$ of B onto a point P_n' of C^3. Since the points P_n' are bounded, we may assure that they converge to a point $P' \neq P$ as $n \to \infty$. (If $P' = P$, then the system would degenerate to a minimal surface spanning C^1C^2, a possibility which we exclude.) By the methods of [2] u_n can be seen to converge to a point $u' \neq \pm 1$, as $n \to \infty$. The convergence of $\{\bar{x}^n\}$ is then proved exactly as in [8], sec. 6. The transversality conditions for the solution dictate that along the image curve of $[u', 1]$ the component surfaces meet at angles of 120°, while along the image curve of $[-1, u']$ the solution surfaces S_1, S_2 are continuations of each other as minimal surfaces.

3. Solution of the problem for other contours

The methods of [8] can be applied without effort to other contours, subject to similar symmetry conditions. Such a contour is that consisting of four Jordan arcs joined at two points, which is spanned by a system of five surfaces. (See Fig. 7). Because of the similarity of this problem to that of Section 1, we do not include the solution.

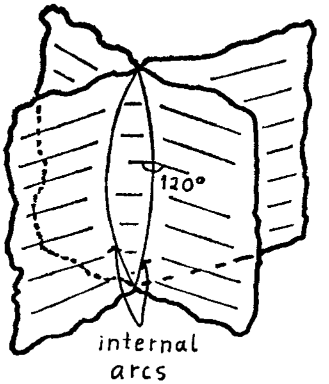

Fig. 7. System of five surfaces.

Appendix

As in [8] we seek an analytic vector function $T(w)$ satisfying the boundary conditions

$$(1)\qquad \begin{aligned} \bar{D}T^{+} + CT^{-} &= 0 \text{ on } L, \\ DT^{+} + \bar{D}T^{-} &= 0 \text{ on } L_t, \end{aligned}$$

with

$$D = \tfrac{1}{2}J \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} J^{-1}, \quad J = \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix},$$

which is of the form

$$T(w) = \tfrac{1}{2}J \begin{bmatrix} T_1(w) & 0 \\ 0 & T_2(w) \end{bmatrix} J^{-1}.$$

Upon substitution into (1) we obtain the piecewise constant solutions

$$\begin{aligned} T_1(w) &= 1/3, \quad \text{for } |w| < t^2, \\ &= -1, \quad \text{for } t^2 < |w| < 1, \\ &= 3, \quad \text{for } |w| > 1, \\ T_2(w) &= 1/T_1(w). \end{aligned}$$

Eq. 10 then follows immediately from [5], p. 236

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